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Duals of Compact Lie Groups Realized in the Cuntz Algebras and Their Actions on C^*-Algebras

SERGIO DOPLICHER*

Dipartimento di Matematica, Università di Roma "La Sapienza," 00185 Rome, Italy

AND

JOHN E. ROBERTS

Universität Osnabrück, Fachbereich Physik, D. 4500 Osnabrück, Federal Republic of Germany

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As a first step towards a new duality theorem for compact groups we consider a representation category $\mathscr{T}_G$ of a compact Lie group $G \subset U(d, \mathbb{C})$ whose objects are the tensor powers of the defining representation and whose arrows are the intertwiners. We associate to $\mathscr{T}_G$ in a natural way a C^*-algebra O_G which can be identified with the fixed points of the Cuntz algebra O_d under the natural action of G. O_G carries a canonical endomorphism σ_G, and $\mathscr{T}_G$ can be reconstructed from $\{O_G, \sigma_G\}$. If $G \subset SU(d)$ then O_G is simple. We give conditions for an endomorphism ρ of a unital C^*-algebra $\mathscr{A}$ to determine an action of $\mathscr{T}_{SU(d)}$ on $\mathscr{A}$. Such actions correspond to $*$-monomorphisms of $O_{SU(d)}$ into $\mathscr{A}$ with natural intertwining properties. © 1987 Academic Press, Inc.

1. INTRODUCTION

In [1] Cuntz showed that isometries ψ_i, $i = 1, 2, ..., d$, satisfying

$$\psi_i^* \psi_j = \delta_{ij} I$$
$$\sum_{i=1}^{d} \psi_i \psi_i^* = I, \tag{1.1}$$

generate a unique C^*-algebra O_d. As we shall, in the course of this paper, be giving a slightly different and more algebraic proof of this result, we begin with the $*$-algebra over the complex numbers $\mathbb{C}$ with unit I generated

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by these elements and relations. This *-algebra will be denoted by ${}^0O_d$ and referred to as the algebraic part of O_d. The linear span of the ψ_i, $i = 1, 2, \dots, d$, will be denoted by H and called the canonical Hilbert space in ${}^0O_d$. The scalar product in H is defined by

$$(\psi, \psi')I = \psi^*\psi'. \tag{1.2}$$

The linear subspace generated by $\psi_{i_1}\psi_{i_2}\cdots\psi_{i_r}$ will be denoted H^r, the rth tensor power of H, and the linear span of terms of the form $\psi_{i_1}\psi_{i_2}\cdots\psi_{i_r}\psi^*_{j_s}\cdots\psi^*_{j_2}\psi^*_{j_1}$ by (H^s, H^r). (H^s, H^r) can be identified with the set of linear mappings from H^s to H^r. We interpret H^0 as $\mathbb{C}I$ in an obvious way. Now, by virtue of (1.1), $(H^s, H^r) \subset (H^{s+1}, H^{r+1})$, and this allows us to define for $k \in \mathbb{Z}$,

$$ {}^0O_d^k = \bigcup_{r, k+r \geqslant 0} (H^r, H^{r+k}) \tag{1.3}$$

and makes ${}^0O_d$ into a $\mathbb{Z}$-graded *-algebra.

The above discussion shows that the *-algebra ${}^0O_d$ is really constructed from the canonical Hilbert space H. An alternative point of view on ${}^0O_d$ bringing out the role of H will prove instructive.

Let H be a Hilbert space of finite dimension $d > 1$ and let $\mathscr{T}_d$ denote the category whose objects are the tensor powers H^r, $r \in \mathbb{N}_0$, where the set (H^s, H^r) of arrows from H^s to H^r are the linear mappings. We can summarize the algebraic structure of $\mathscr{T}_d$ and the norm on the arrows by saying that $\mathscr{T}_d$ is a C^*-category. Moreover, since the objects of $\mathscr{T}_d$ are the tensor powers of a given Hilbert space H, the tensor product is defined within $\mathscr{T}_d$ and is strictly associative. We refer to the structure obtained in this way as a strict monoidal C^*-category.

Now ${}^0O_d$, and hence O_d, can be derived from $\mathscr{T}_d$ by a very simple construction. Comparing the two structures, we note that the relation $(H^s, H^r) \subset (H^{s+1}, H^{r+1})$ is valid in ${}^0O_d$ but not in $\mathscr{T}_d$. However, the mappings $S \to S \otimes 1$ from (H^s, H^r) to (H^{s+1}, H^{r+1}) are injective and if we *define* ${}^0O_d^k$ to be the inductive limit of

$$(H^r, H^{r+k}) \xrightarrow{\otimes 1} (H^{r+1}, H^{r+1+k}) \xrightarrow{\otimes 1} \cdots \xrightarrow{\otimes 1} (H^{r+n}, H^{r+n+k}) \xrightarrow{\otimes 1} \cdots$$

then the product of an $R \in {}^0O_d^j$ and an $S \in {}^0O_d^k$ can always be defined as an element of ${}^0O_d^{j+k}$ since, for r sufficiently large, $S \in (H^r, H^{r+k})$, $R \in (H^{r+k}, H^{r+k+j})$, and $(R \otimes 1) \circ (S \otimes 1) = (R \circ S) \otimes 1$, where $\circ$ denotes the composition in $\mathscr{T}_d$. Furthermore, S^* is well defined as an element of ${}^0O_d^{-k}$ since $(S \otimes 1)^* = S^* \otimes 1$. Thus setting ${}^0O_d = \oplus_{r \in \mathbb{Z}} {}^0O_d^r$ gives us a $\mathbb{Z}$-graded *-algebra. Picking $\psi_i \in (\mathbb{C}, H)$ such that $\psi_i 1$, $i = 1, 2, \dots, d$, is an orthonormal basis of H, we verify (1.1) and check that the ψ_i generate ${}^0O_d$ as a *-algebra.

The operation of tensoring on the right by $1 \in (H, H)$ in $\mathcal{T}_d$ has been used as an identification map in constructing ${}^0\mathcal{O}_d$ so that, by definition, it induces the identity automorphism of ${}^0\mathcal{O}_d$. However, the operation of tensoring on the left by $1 \in (H, H)$ in $\mathcal{T}_d$ induces a non-trivial endomorphism σ of the $\mathbb{Z}$-graded *-algebra ${}^0\mathcal{O}_d$ called the canonical endomorphism. Since

$$R \otimes 1 \circ 1 \otimes S = R \otimes S = 1 \otimes S \circ R \otimes 1$$

in $\mathcal{T}_d$, we have the relation

$$\psi C = \sigma(C)\psi, \qquad \psi \in (\mathbb{C}, H), \quad C \in {}^0\mathcal{O}_d.$$

More generally, if $R \in (H^s, H^r)$ then

$$R\sigma^s(C) = \sigma^r(C)R, \qquad C \in {}^0\mathcal{O}_d.$$

Note, by (1.1), that the canonical endomorphism can be expressed as

$$\sigma(C) = \sum_{i=1}^{d} \psi_i C \psi_i^*, \qquad C \in {}^0\mathcal{O}_d.$$

An immediate benefit of this alternative construction of ${}^0\mathcal{O}_d$ is that it can be generalized. Let G be a (closed) subgroup of the unitary group $U(H)$ of H, then H^r carries a natural unitary representation of G namely the rth tensor power of the defining representation. We now consider the category $\mathcal{T}_G$ whose objects are the G-modules H^r, $r \in \mathbb{N}_0$ and where the arrows are the G-module homomorphisms. We denote the set of arrows from H^s to H^r in $\mathcal{T}_G$ by $(H^s, H^r)_G$. $\mathcal{T}_G$ is again a strict monoidal C^*-category and we can repeat the above construction defining ${}^0\mathcal{O}_G^k$ to be the inductive limit of

$$(H^r, H^{r+k})_G \xrightarrow{\otimes 1} (H^{r+1}, H^{r+1+k})_G \xrightarrow{\otimes 1} \cdots \xrightarrow{\otimes 1} (H^{r+n}, H^{r+n+k})_G \xrightarrow{\otimes 1} \cdots$$

to get a $\mathbb{Z}$-graded *-algebra ${}^0\mathcal{O}_G = \bigoplus_{k \in \mathbb{Z}} {}^0\mathcal{O}_G^k$ carrying an endomorphism σ_G derived from the operation of tensoring on the left by $1 \in (H, H)_G$.

Of course, $\mathcal{T}_G$ is just the fixed-point subcategory of $\mathcal{T}_d$ under the adjoint action of G so that ${}^0\mathcal{O}_G$ is nothing but the fixed-point subalgebra of ${}^0\mathcal{O}_d$ under the induced action of the group G by $\mathbb{Z}$-graded *-automorphisms. If we were merely to adopt this definition we would miss the essential point that ${}^0\mathcal{O}_G$ is determined by $\mathcal{T}_G$ as an abstract strict monoidal C^*-category whereas, if we want ${}^0\mathcal{O}_G$ to be a subalgebra of ${}^0\mathcal{O}_d$, we need $\mathcal{T}_G$ as a concrete category, i.e., as a subcategory of $\mathcal{T}_d$. This remark and the results announced in [2] lead to a new duality theory for compact groups which strengthens the Tannaka–Krein duality theorem by characterizing the abstract category of representations of a compact group and will be discussed elsewhere (cf. also [12, 13]).

Here we first collect together some results on the Cuntz algebras O_d and their fixed-point algebras O_G derived in the course of this work and of independent interest. Section 2 studies the grading and ideal structure of the algebraic parts of these algebras and culminates in the result (Theorem 2.12) that if G is a subgroup of the special unitary group of H, the group of unitaries of determinant one, then ${}^0O_G$ admits a unique C^*-seminorm which is in fact a C^*-norm, so that the completion O_G is a simple C^*-algebra. In Section 3, we study the C^*-algebras O_G and their canonical endomorphisms proving in particular (Theorem 3.5) that all intertwiners between powers of σ_G are algebraic. We then apply these results to endomorphisms of unital C^*-algebras with permutation symmetry in Section 4. We classify the permutation symmetries for irreducible endomorphisms with a left inverse on a unital C^*-algebra with trivial centre (Theorem 4.3) and take a first step towards proving the results announced in [2] by giving conditions (Corollary 4.4) permitting one to define an action of $O_{SU(d)}$ on a C^*-algebra.

2. Grading and C^*-Seminorms

We begin our discussion of the $\mathbb{Z}$-graded *-algebras ${}^0O_G$ with some remarks on permutations. The unitary operator

$$\theta(r, s)\colon \quad H^r \otimes H^s \to H^s \otimes H^r$$

permuting the order of the factors in the tensor product is in $(H^{r+s}, H^{r+s})_G$ for any subgroup G of $U(H)$. These operators satisfy

$$\theta(r, s) \circ \theta(s, r) = 1_{r+s} \tag{2.1}$$

$$1_s \otimes \theta(r, t) \circ \theta(r, s) \otimes 1_t = \theta(r, s+t) \tag{2.2}$$

$$\theta(r, 0) = \theta(0, r) = 1_r, \tag{2.3}$$

where 1_r denotes the unit of $(H^r, H^r)_G$. Furthermore

$$\theta(r, r') \circ R \otimes R' = R' \otimes R \circ \theta(s, s'), \qquad R \in (H^s, H^r)_G,\ R' \in (H^{s'}, H^{r'})_G. \tag{2.4}$$

This additional structure on $\mathcal{T}_G$ makes it into a strict symmetric monoidal C^*-category.

The images of the unitary operators $\theta(r, s)$ in ${}^0O_G$, denoted by the same symbol, generate a unitary representation $p \to \theta(p)$ of $\mathbb{P}_\infty$, the group of finite permutations of the integers $\mathbb{N}$, in ${}^0O_G^0$. $\mathbb{P}_\infty$ is the inductive limit of the subgroups $\mathbb{P}_n$ leaving $n+1$, $n+2$,... fixed. $\theta(r, s)$ implements the permutation $p = \left(\begin{smallmatrix} 1 & 2 & \cdots & r+s \\ s+1 & s+2 & \cdots s+r & 1\ 2 \cdots s \end{smallmatrix}\right) \in \mathbb{P}_{r+s}$. If we abuse notation by using the symbol σ to denote the canonical endomorphism on ${}^0O_G$ as well as the endomorphism of $\mathbb{P}_\infty$ which shifts to the right

$$\sigma p(1) = 1, \qquad (\sigma p)(n) = 1 + p(n-1), \quad n > 1,\ p \in \mathbb{P}_\infty$$

then

$$\theta(\sigma p) = \sigma(\theta(p)), \qquad p \in \mathbb{P}_\infty. \tag{2.5}$$

If we write θ for $\theta(1, 1)$ we see that $\theta(p)$ for any non-trivial permutation $p \in \mathbb{P}_n$ is a word in $\theta, \sigma(\theta),..., \sigma^{n-2}(\theta)$. In particular,

$$\theta(r, 1) = \theta\sigma(\theta) \cdots \sigma^{r-1}(\theta). \tag{2.6}$$

Finally, for computational purposes it is useful to note that in terms of the generators of ${}^0O_d$ we have

$$\theta(p) = \sum_{i_1, i_2, ..., i_n} \psi_{i_1}\psi_{i_2} \cdots \psi_{i_n}\psi^*_{i_{p(n)}} \cdots \psi^*_{i_{p(1)}}. \tag{2.7}$$

In terms of ${}^0O_G$, the essential content of (2.4) can be summarized in the equation

$$\theta(r, 1)R = \sigma(R)\,\theta(s, 1), \qquad R \in (H^s, H^r)_G. \tag{2.8}$$

We conclude that

$$\sigma(R) = \lim_{r \to \infty} \theta(r+k, 1)\,R\theta(r, 1)^*, \qquad R \in {}^0O^k_G \tag{2.9}$$

in the sense that this equation holds for all sufficiently large r.

This equation provides a starting point for our investigation of ideals in ${}^0O_G$. An ideal I in ${}^0O_G$ will be said to be graded if

$$I = \bigoplus_{k \in \mathbb{Z}} I \cap {}^0O^k_G.$$

As a consequence of (2.9) we have

2.1. LEMMA. *Any graded ideal in ${}^0O_G$ is σ-stable.*

As we shall see, the assumption that $G \subset SU(H)$, the subgroup of unitaries of H of determinant one, has a decisive influence on the ideal structure of ${}^0O_G$. When $G \subset SU(H)$ then G acts trivially on the totally antisymmetric subspace of H^d. Hence there is an isometry S in $(\mathbb{C}, H^d)_G$

unique up to a phase whose range is the totally antisymmetric subspace of H^d. We can fix this phase relative to the chosen basis in H by setting

$$S = \frac{1}{\sqrt{d!}} \sum_{p \in \mathbb{P}_d} \operatorname{sign} p \; \psi_{p(1)} \cdots \psi_{p(d)}. \tag{2.10}$$

We can obviously write

$$S = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \psi_i \hat{\psi}_i, \tag{2.11}$$

$$\hat{\psi}_i = \frac{1}{\sqrt{(d-1)!}} \sum_{p \in \mathbb{P}_d(i)} \operatorname{sign} p \; \psi_{p(2)} \cdots \psi_{p(d)}, \tag{2.12}$$

where $\mathbb{P}_d(i)$ denotes the subset of $\mathbb{P}_d$ of permutations with $p(1) = i$. We note for future reference the following simple lemma which can be proved by direct computation.

2.2. LEMMA. $S^*\sigma(S) = (-1)^{d-1}(1/d)I$

$$\psi_i^* = (-1)^{d-1} \sqrt{d}\, S^* \hat{\psi}_i$$

$$\hat{\psi}_i^* \hat{\psi}_j = \delta_{ij} I.$$

Now the unitary transformations on H^m induced by the elements of G also have determinant one. Consequently, we have

2.3. COROLLARY. *If $G \subset SU(H)$ there are isometries $S_m \in (\mathbb{C}, H^{md^m})_G$, $m \in \mathbb{N}$ with*

$$S_m^* \sigma^m(S_m) = (-1)^{d^m - 1} \frac{1}{d^m} I \equiv \lambda_m I.$$

The isometries S_m are used in the following computation: suppose

$$X = X_0 + \left(\sum_{k=-n}^{-m} + \sum_{k=m}^{n} \right) X_k, \tag{2.13}$$

where $X_k \in {}^0O_G^k$. Then for some sufficiently large integer $r \geq m$, $X_k \in (H^r, H^{r+k})_G$, $-n \leq k \leq n$. Hence

$$\sigma^r(S_m^*)\, X \sigma^r(S_m) = \sigma^r(S_m^*) \left[\sigma^r(S_m)\, X_0 + \left(\sum_{k=-n}^{-m} + \sum_{k=m}^{n} \right) \sigma^{r+k}(S_m)\, X_k \right].$$

The coefficient of X_{-m} is $\sigma^{r-m}(\sigma^m(S_m^*)S_m) = \lambda_m I$ and of X_m is $\sigma^r(S_m^*\sigma^m(S_m)) = \lambda_m I$. Hence defining

$$X' \equiv \frac{1}{1-\lambda_m}\,(\sigma^r(S_m^*)\,X\sigma^r(S_m) - \lambda_m X), \tag{2.14}$$

we see that

$$X' = X_0 + \left(\sum_{k=-n}^{-(m+1)} + \sum_{k=m+1}^{n}\right) X_k', \tag{2.15}$$

where

$$X_k' = \frac{1}{1-\lambda_m}\,(\sigma^r(S_m^*)\,\sigma^{r+k}(S_m) - \lambda_m)\,X_k \in {}^0O_G^k.$$

We use this computation to investigate ideals which are the kernels of Hilbert space representations of ${}^0O_G$, i.e., *-homomorphisms of ${}^0O_G$ into the algebra of all bounded operators on some Hilbert space or equivalently of *-homomorphisms into some C^*-algebra.

2.4. Lemma. *If $G \subset SU(H)$ and π is a Hilbert space representation of ${}^0O_G$ then* $\ker \pi$ *is a graded ideal.*

Proof. If the X of (2.13) is in $\ker \pi$ then so is the X' of (2.15) and we conclude by iteration that $\pi(X_0) = 0$. But $\pi(X) = 0$ implies $\pi(X^*X) = 0$ and hence, taking the component of zero grade, that $\pi(\sum_k X_k^* X_k) = 0$. Thus $\pi(X_k) = 0$ and $\ker \pi$ is graded.

2.5. Lemma. *If $G \subset SU(H)$, a Hilbert space representation π of ${}^0O_G$ is faithful if and only if its restriction to ${}^0O_G^0$ is faithful.*

Proof. Suppose $\pi \mid {}^0O_G^0$ is faithful, $X \in {}^0O_G^k$ and $\pi(X) = 0$. Then $\pi(X^*X) = 0$ but $X^*X \in {}^0O_G^0$ so $X^*X = 0$ which implies that $X = 0$. Since, by Lemma 2.4, $\ker \pi$ is graded; the result follows.

Since ${}^0O_d^0$ is an inductive limit of full matrix algebras we conclude

2.6. Corollary. *Every Hilbert space representation of ${}^0O_d$ is faithful.*

We want to extend this last result to ${}^0O_G$ for $G \subset SU(H)$ and begin with a lemma on the conditional expectation m from ${}^0O_d$ to ${}^0O_G$ obtained by averaging over the group.

2.7. Lemma. *If $G \subset SU(H)$ then m: ${}^0O_d \to {}^0O_G$ is completely positive and faithful.*

Proof. Since every element C of ${}^0O_d$ is G-finite, i.e., is an element of some finite-dimensional G-invariant subspace, we can, by choosing a suitable basis in that subspace, write $C = \sum_r \sum_{k=1}^{m_r} \sum_{s=1}^{d_r} \lambda^r_{ks} C^r_{ks}$, where r labels the irreducible representations occurring in C, d_r denote their dimensions, and m_r their multiplicities, and

$$\alpha_g(C^r_{ks}) = \sum_{s'} C^r_{ks'} u^r(g)_{s's}$$

$$\int \overline{u^{r'}(g)_{s's}} u^r(g)_{t't}\, d\mu(g) = \frac{1}{d_r} \delta_{s't'} \delta_{st} \delta_{r'r}.$$

Hence $m(C^*C) = \int \alpha_g(C^*C)\, d\mu(g) = \sum_{r,s,s'} B^{r*}_{s's} B^r_{s's}$, where $B^r_{s's} = (1/\sqrt{d_r}) \sum_k \lambda^r_{ks} C^r_{ks'}$ transforms irreducibly under G. Thus m is faithful and $m(C^*C)$ is positive in ${}^0O_d$. To show that it is actually positive in ${}^0O_G$ it suffices to show that $\sum C^*_k C_k$ is positive in ${}^0O_G$ when the C_k, say $k = 1, 2,..., 1$, transform irreducibly under G. But then since $G \subset SU(H)$ we can find S_k, $k = 1, 2,..., l$ in $(\mathbb{C}, H^r)$ for some r with $S^*_j S_k = \delta_{jk}$ which transform as C^*_k under G. Hence $Y = \sum_j S_j C_j \in {}^0O_G$ and $C_j = S^*_j Y$. Thus $\sum_j C^*_j C_j = Y^* \sum_j S_j S^*_j Y$. But $\sum_j S_j S^*_j$ being a G-invariant projection is obviously positive in ${}^0O_G$ and hence so is $\sum_j C^*_j C_j$. We have now shown that m is positive using two properties of ${}^0O_d$: first that every element is G-finite and second that there is a multiplet of isometries in ${}^0O_d$ transforming according to any given irreducible representation. Thus the same proof applied to the *-algebra of $n \times n$-matrices over ${}^0O_d$ shows that m is completely positive.

An obvious consequence of this lemma is that any element of ${}^0O_G$ which is positive as an element of ${}^0O_d$ is also positive as an element of ${}^0O_G$. We shall use this lemma to be able to induce up Hilbert space representations from ${}^0O_G$ to ${}^0O_d$.

2.8. Theorem. *Let $G \subset SU(H)$ then any Hilbert space representation π of ${}^0O_G$ is faithful.*

Proof. We induce π up to a representation $\tilde{\pi}$ of ${}^0O_d$ using the conditional expectation m: ${}^0O_d \to {}^0O_G$. Vectors of the form $C \otimes \Phi$ with $C \in {}^0O_d$ and $\Phi \in H_\pi$ are total in the representation space $H_{\tilde{\pi}}$ of $\tilde{\pi}$ and

$$(C \otimes \Phi, C' \otimes \Phi') = (\Phi, \pi m(C^*C')\Phi').$$

We thus have $C \otimes \pi(X)\Phi = CX \otimes \Phi$, $X \in {}^0O_G$. We claim that $\pi(X) = 0$ implies $\tilde{\pi}(X) = 0$. It suffices to show that

$$\tilde{\pi}(X)\psi_{i_1} \cdots \psi_{i_r} \psi^*_{j_s} \cdots \psi^*_{j_1} \otimes \Phi = 0$$

for all choices of $i_1,\ldots, i_r, j_1,\ldots, j_s$. Now, by Lemma 2.2,

$$\psi_i = (-1)^{d-1}\sqrt{d}\,\hat{\psi}_i^* S$$

with $S \in {}^0O_G$ since $G \subset SU(H)$. Thus

$$X\psi_i = (-1)^{d-1}\sqrt{d}\,\hat{\psi}_i^* \sigma^{d-1}(X) S.$$

Hence $X\psi_{i_1}\cdots\psi_{i_r}\psi_{j_s}^*\cdots\psi_{j_1}^* = C\sigma^{r(d-1)+s}(X)T$, where $C \in {}^0O_d$ and $T \in {}^0O_G$. Thus

$$\tilde{\pi}(X)\psi_{i_1}\cdots\psi_{i_r}\psi_{j_s}^*\cdots\psi_{j_1}^* \otimes \Phi = C \otimes \pi(\sigma^{r(d-1)+s}(X)T)\Phi = 0$$

since $\ker \pi$ is σ-stable by Lemmas 2.4 and 2.1. But $\tilde{\pi}(X) = 0$ implies $X = 0$ by Corollary 2.6 completing the proof.

As every C^*-seminorm on a $*$-algebra has the form $\|X\| = \|\pi(X)\|$ for some Hilbert space representation π we conclude

2.9. Corollary. *If $G \subset SU(H)$, every C^*-seminorm on ${}^0O_G$ is actually a C^*-norm.*

The next goal is to show that ${}^0O_G$ has a unique C^*-norm whenever $G \subset SU(H)$. Of course ${}^0O_G^0$ being an inductive limit of C^*-algebras does have a unique C^*-norm. As ${}^0O_G$ is a $\mathbb{Z}$-graded $*$-algebra we have a canonical action of $\mathbb{T}$ by automorphisms of ${}^0O_G$ with

$$\alpha_\lambda(X) = \lambda^k X, \qquad X \in {}^0O_G^k$$

and the projection $m_0 \colon {}^0O_G \to {}^0O_G^0$ is just the conditional expectation gotten by integating over the circle group with respect to the normalized Haar measure.

Now if $\|\ \|$ is a C^*-norm on ${}^0O_G$ and X is as in (2.13), we conclude from (2.14) that

$$\|X'\| \leqslant \frac{1+(1/d)^m}{1-(1/d)^m}\|X\|$$

since S_m is isometric. Now in view of (2.15) we see that if $X = \sum_{k=-n}^n X_k$, $X_k \in {}^0O_G^k$, then

$$\|m_0(X)\| = \|X_0\| \leqslant \prod_{m=1}^n \frac{1+(1/d)^m}{1-(1/d)^m}\|X\|.$$

Since

$$\prod_{m=1}^\infty \frac{1+(1/d)^m}{1-(1/d)^m} < +\infty,$$

we have

2.10. LEMMA. *If $G \subset SU(H)$ then the projection m_0 onto the grade zero part of ${}^0O_G$ is continuous for any C^*-norm on ${}^0O_G$.*

In fact, it is easy to see directly that $\|X_k\| \leqslant \|X\|$. At this point, the argument given by Cuntz [1, 1.10] going back to [3, 1.2.5] and employing Fourier analysis can be used to show the uniqueness of the C^*-norm on ${}^0O_G$. We provide an alternative argument based on the conditional expectation m_0 and, for future reference, replace the circle group by a general compact group.

2.11. LEMMA. *Let $\mathscr{B}$ be a *-algebra and α an action of a compact group G by automorphisms of $\mathscr{B}$. Suppose every element of $\mathscr{B}$ is G-finite then there is at most one C^*-norm on $\mathscr{B}$ extending a given C^*-norm on the fixed-point algebra $\mathscr{A}$ such that either*

(a) $\|\alpha_g(B)\| = \|B\|$, $g \in G$, $B \in \mathscr{B}$ *or*

(b) $m: \mathscr{B} \to \mathscr{A}$ *is continuous,*

where m is the conditional expectation derived by integrating the action with respect to the normalized Haar measure

$$m(B) = \int \alpha_g(B)\, d\mu(g).$$

Proof. Any norm on $\mathscr{B}$ satisfying (a) automatically satisfies (b) since $\|m(B)\| \leqslant \int \|\alpha_g(B)\|\, d\mu(g) = \|B\|$.

Now let $\mathscr{C}$ be the completion of $\mathscr{B}$ in a C^*-norm extending the given norm on $\mathscr{A}$ and satisfying (b). The maximal C^*-norm on $\mathscr{B}$ coinciding with the given norm on $\mathscr{A}$ obviously satisfies (a) and if $\mathscr{L}$ is the completion of $\mathscr{B}$ in this norm we have a homomorphism $\pi: \mathscr{L} \to \mathscr{C}$ which is the identity on $\mathscr{B}$. In particular, $m(X^*X) = m \circ \pi(X^*X)$ for $X \in \mathscr{B}$. But since m and $m \circ \pi$ are continuous, this equation holds in the completion of $\mathscr{A}$ for each $X \in \mathscr{L}$. Thus $\pi(X) = 0$ implies $m(X^*X) = 0$ and hence $X = 0$ since the mean over a compact group acting continuously on a C^*-algebra is always a faithful conditional expectation. Then π is injective completing the proof.

As ${}^0O_d$ has a Hilbert space representation we do not need to invoke any general existence theorems at this point to show that ${}^0O_G$ has a C^*-norm. We have therefore proved

2.12. THEOREM. *If $G \subset SU(H)$ then there is a unique C^*-seminorm on ${}^0O_G$ which is actually a C^*-norm.*

3. The C^*-Algebras O_G

In this section we discuss the C^*-algebras O_G determined canonically by the representation theory of a closed subgroup G of the unitary group $U(H)$ of H on the tensor powers of H. We can define O_G to be the completion of ${}^0O_G$ in the maximal C^*-norm. By Lemma 2.11, this is the unique C^*-norm on ${}^0O_G$ for which the projection $m_0: {}^0O_G \to {}^0O_G^0$ onto the grade zero subalgebra is continuous. We write $O_{SU(d)}$ and $O_{U(d)}$ to denote the C^*-algebras obtained by taking $G = SU(H)$ and $G = U(H)$, respectively, so as to emphasize the dimension involved. When G reduces to the identity we get the Cuntz algebra O_d. O_G is just the fixed point subalgebra under the natural action of G on O_d.

If G is a subgroup of $SU(H)$, then by Theorem 2.12, O_G is the completion of ${}^0O_G$ in the unique C^*-seminorm. Thus we have

3.1. Theorem. *If $G \subset SU(H)$ then O_G is a simple C^*-algebra.*

The canonical action of $\mathbb{T}$ on ${}^0O_G$ determined by the grading extends to a continuous action α of $\mathbb{T}$ making O_G into a $\mathbb{Z}$-graded C^*-algebra. The subspace of grade k will be denoted O_G^k,

$$O_G^k = \{X \in O_G : \alpha_\lambda(X) = \lambda^k X\}$$

and is the closure of ${}^0O_G^k$.

Similarly, the canonical endomorphism σ_G of ${}^0O_G$ extends to a canonical endomorphism of O_G also denoted σ_G or, more usually, when G is clear from the context, simply by σ.

Our point of view will be that one should study not O_G alone but the pair $\{O_G, \sigma_G\}$. Indeed, as we shall show, the category $\mathscr{T}_G$ and the $\mathbb{Z}$-graded $*$-algebra ${}^0O_G$ can be recovered from the pair $\{O_G, \sigma_G\}$. We begin with a simple lemma which has been obtained independently by Bratteli and Evans [4, Theorem 3.2].

3.2. Lemma. *Suppose $X \in O_d$ and $X\theta(p) = \theta(p)X$, $p \in \mathbb{P}_\infty$; then $X \in \mathbb{C}I$.*

Proof. Since $\theta(p)$ is invariant under the action $\lambda \to \alpha_\lambda$, it suffices to suppose $\alpha_\lambda(X) = \lambda^k X$, i.e., $X \in O_d^k$. If $k = 0$, then by (2.9),

$$\sigma(X) = \lim_{r \to \infty} \theta(r, 1)\, X\theta(r, 1)^* = X.$$

Thus $\psi X = \sigma(X)\psi = X\psi$, $\psi \in H$ so X is in the centre of the simple C^*-algebra O_d. Hence $X \in \mathbb{C}I$. Suppose $k \neq 0$, then we must show $X = 0$. If not $X^*X = XX^* \in \mathbb{C}I$ by the result for $k = 0$ so X is invertible. But now by (2.9)

$$\sigma(X) = \lim_{r \to \infty} \theta(r + k, 1)\, X\theta(1, r).$$

Thus $X^{-1}\sigma(X)=\lim_{r\to\infty}\theta(r+k,1)\,\theta(1,r)$. But $\|\theta(p)-I\|\geqslant\sqrt{2}$ for $p\neq e$ so the right-hand side is not norm convergent. Hence $X=0$ completing the proof.

3.3. COROLLARY. *O_G has trivial relative commutant in O_d and every automorphism α of O_d leaving O_G pointwise invariant is in α_G.*

Proof. $\psi^*\alpha(\psi')$ is in the relative commutant of O_G for $\psi, \psi'\in H$. Hence $\alpha(H)=H$ and α is induced by a unitary of H. The group K say of unitaries on H which induce automorphisms of O_d leaving O_G pointwise invariant is a compact group that might a priori be larger than G. If it were strictly larger, we could find an $f\in L^2(K)$ which is not a constant but which is invariant under the action of the closed subgroup G. Decomposing into irreducible components we see that in some irreducible representation of K there must be a vector invariant under G but not under K. Every irreducible representation of K may be realized within O_d in fact as a subspace of H^nS^{*k} for some n and k, where S is defined by (2.10). We now have a contradiction since we have found an $X\in O_G$ which is not invariant under K. We are just proving a variant of the Tannaka duality theorem, see, e.g., [5, p. 176] and [6, Appendix C].

3.4. LEMMA. *If $X\in O_d$ and*

$$X\sigma^{s+k}(\theta)=\sigma^{r+k}(\theta)X, \qquad k\in\mathbb{N}_0,$$

then $X\in(H^s, H^r)$.

Proof. $\psi_{i_r}^*\cdots\psi_{i_1}^*X\psi_{j_1}\cdots\psi_{j_s}\sigma^k(\theta) = \sigma^k(\theta)\,\psi_{i_r}^*\cdots\psi_{i_1}^*X\psi_{j_1}\cdots\psi_{j_s}$, $k\in\mathbb{N}_0$. Hence, by Lemma 3.2, $\psi_{i_r}^*\cdots\psi_{i_1}^*X\psi_{j_1}\cdots\psi_{j_s}\in\mathbb{C}I$ so

$$X=\sum_{\substack{i_1\cdots i_r\\ j_1\cdots j_s}}\psi_{i_1}\cdots\psi_{i_r}\psi_{i_r}^*\cdots\psi_{i_1}^*X\psi_{j_1}\cdots\psi_{j_s}\psi_{j_s}^*\cdots\psi_{j_1}^*\in(H^s, H^r).$$

We deduce at once the following result characterizing $\mathcal{T}_G$ as a category of intertwiners for powers of the canonical endomorphism within O_G; all such intertwiners are "algebraic."

3.5. THEOREM. *If $X\in O_G$, then the following conditions are equivalent*

(a) $X\in(H^s, H^r)_G$,

(b) $X\sigma^s(Y)=\sigma^r(Y)X$, $Y\in O_G$.

This result serves as a starting point for the discussion of endomorphisms of C^*-algebras in the next section, where we also need the characterizations of $O_{U(d)}$ and $O_{SU(d)}$ given below.

3.6. LEMMA. *$O_{U(d)}$ is generated as a C^*-algebra by $\theta(p)$, $p \in \mathbb{P}_\infty$. It is therefore the smallest σ-stable C^*-subalgebra of O_d containing θ.*

Proof. $(H^r, H^s)_{U(d)} = 0$ if $r \neq s$ as follows by looking at the representation of the centre $\mathbb{T}$ of $U(d)$. $(H^r, H^r)_{U(d)}$ is the algebra generated by the unitaries permuting the order of the factors in the tensor power cf. [7, Theorem 4.4.E]. Thus $O_{U(d)}$ is the C^*-algebra generated by $\theta(p)$, $p \in \mathbb{P}_\infty$. This has been noted independently in [4].

Thus $O_{U(d)}$ is just a quotient of $C^*(\mathbb{P}_\infty)$, the C^*-algebra of the group $\mathbb{P}_\infty$, by some ideal I_d. $C^*(\mathbb{P}_\infty)$ is the inductive limit of the finite-dimensional algebras $C^*(\mathbb{P}_n)$, so $O_{U(d)}$ is determined by specifying $I_d \cap C^*(\mathbb{P}_n)$ for each n, or equivalently by the quasiequivalence class of the representation of each $\mathbb{P}_n$. The quasiequivalence class of $p \to \theta(p)$, $p \in \mathbb{P}_n$ is obviously the class of the natural representation on H^n and corresponds to the set of Young diagrams of n squares and at most d rows. Incidentally, looking at $O_{U(d)}$ in this way as an inductive limit of finite-dimensional algebras shows clearly that it has many ideals in contrast to the simple C^*-algebras $O_{SU(d)}$. In fact $I_d \subset I_{d'}$ when $d' < d$ so that $O_{U(d')}$ is a quotient of $O_{U(d)}$ for $d' < d$.

The scalar product of two vectors is the basic invariant for $U(d)$ and we see from (2.7) that $\theta(p)$ is indeed built up of scalar products. The basic invariants for $SU(d)$ are the scalar product and the determinant so a glance at (2.10) suggests the following result.

3.7. LEMMA. *$O_{SU(d)}$ is the smallest σ-stable C^*-subalgebra of O_d containing θ and S.*

Proof. Let α denote the natural action of the unitary group of H, i.e., α_U for $U \in (H, H)$ is the automorphism of O_d with $\alpha_U(\psi) = U\psi$. A computation using (2.10) shows that $\alpha_U(S) = S \det U$ so $S \in O_{SU(d)}$. Since $U(d)$ is generated by $SU(d)$ and its centre $\mathbb{T}$, $O^0_{SU(d)} = O_{U(d)}$. Since $SU(d) \cap \mathbb{T} = \{\lambda \in \mathbb{T}; \lambda^d = 1\}$, $O^k_{SU(d)} = \{0\}$ unless $k = 0 \bmod d$. If $X \in O^{nd}_{SU(d)}$ then $X = XS^{*n}S^n$. But since $S \in O^d_{SU(d)}$, $XS^{*n} \in O^0_{SU(d)}$. As $O_{SU(d)}$ is σ-stable, the result now follows from Lemma 3.6.

The proof of this lemma makes it clear that $O_{SU(d)}$ is constructed in a simple way from $O_{U(d)}$ and the isometry S. In fact, S induces an isomorphism τ of $O_{U(d)}$ onto a subalgebra

$$\tau(A) = SAS^* = \sigma^d(A)\, SS^* = SS^* \sigma^d(A), \qquad A \in O_{U(d)}$$

$$\tau(I) = SS^* = \frac{1}{d!} \sum_{p \in \mathbb{P}_d} \operatorname{sign}(p)\, \theta(p).$$

The range of τ is the subalgebra $\tau(I)\,O_{U(d)}\tau(I)$ and $\tau^{-1}(B)=S^*BS$ for $B\in\tau(O_{U(d)})$, since $\tau(S^*BS)=SS^*BSS^*=B$, $B\in\tau(I)\,O_{U(d)}\tau(I)$; $S^*\tau(A)S=S^*SAS^*S=A$, $A\in O_{U(d)}$.

Now $O_{SU(d)}$ is the cross-product of $O_{U(d)}$ by the action of τ. Indeed the structure of $O_{SU(d)}$ as a cross-product is directly manifested through the grading since an element of $O^{kd}_{SU(d)}$ for $k\geqslant 0$ can be written uniquely in the form AS^k with $A=A\tau^k(I)$ and we have, for example,

$$AS^kBS^l=A\tau^k(B)\,S^{k+l}$$

$$S^{k*}A^*CS^{k+r}=\tau^{-k}(A^*C)S^r$$

for $B=B\tau^l(I)$ and $C=C\tau^{k+r}(I)$. The universal property of the cross product now yields.

3.8. Lemma. *Let $\mu_0\colon O_{U(d)}\to\mathscr{A}$ be a morphism of unital C^*-algebras and $R\in\mathscr{A}$ and an isometry of $\mathscr{A}$ with*

$$R\mu_0(A)R^*=\mu_0(\tau(A)),\qquad A\in O_{U(d)},$$

then μ_0 extends uniquely to a morphism $\mu\colon O_{SU(d)}\to\mathscr{A}$ with $\mu(S)=R$.

We conclude this section with a brief discussion of the behaviour of the correspondence $G\to O_G$ on passing to a quotient group. Given a closed subgroup G of $U(H)$ and an isometry $W\colon H'\to H$ with $WW^*=E\in(H,H)_G$ then there is a closed subgroup G' of $U(H')$ and an epimorphism $q\colon G\to G'$ defined by $q(g)=W^*gW$.

Suppose first that G reduces to the unit of $U(H)$ and letting $d=\dim H$, $d'=\dim H'$, where we suppose $d'>1$, O_G and $O_{G'}$ reduce to O_d and $O_{d'}$. Let $\mathscr{E}$ denote the C^*-subalgebra with unit generated by $EH\subset O_d$, and let $\mathscr{K}$ denote the closed ideal in $\mathscr{E}$ generated by $(I-E)$. Then, as is easily shown [1, Proposition 3.1], $\mathscr{K}$ is the C^*-algebra of compact operators and the quotient, being a C^*-algebra generated by a Hilbert space of dimension d', is $O_{d'}$. Thus the map $\psi\to W^*\psi$, $\psi\in EH$ extends to an epimorphism of $\mathscr{E}$ onto $O_{d'}$ and we have an exact sequence

$$0\to\mathscr{K}\to\mathscr{E}\to O_{d'}\to 0.$$

Now when $E\in(H,H)_G$ for some $G\subset U(H)$, both $\mathscr{E}$ and $\mathscr{K}$ are stable under α_G. In particular $O_G\cap\mathscr{E}$ is the fixed-point subalgebra of $\mathscr{E}$ under α_G and is hence generated by the subsets $E^{\otimes s}(H^r,H^s)_GE^{\otimes r}$, $r,s\in\mathbb{N}_0$. The image of such a subset in $O_{d'}$ is $W^{\otimes s}(H^r,H^s)_GW^{\otimes r}=(H'^r,H'^s)_{G'}$ as follows from $G'=W^*GW$ by taking tensor products. These images generate $O_{G'}$ and, bearing Corollary 3.3 in mind we have shown

3.9. LEMMA. *In the above exact sequence, the image of $O_G \cap \mathscr{E}$ is canonically isomorphic to $O_{G'}$. Every automorphism of $O_{d'}$ leaving $O_{G'}$ pointwise fixed is actually induced by an element of $\alpha_G \mid \mathscr{E}$.*

4. ENDOMORPHISMS WITH PERMUTATION SYMMETRY

We work with the category of C^*-algebras with unit; all morphisms will preserve the unit. If $\mathscr{A}$ is such an algebra then End $\mathscr{A}$ denotes the C^*-category whose objects are the endomorphisms of $\mathscr{A}$ and where the set (ρ, ρ') of arrows from ρ to ρ' are the intertwiners

$$R \in (\rho, \rho') \quad \text{if } R \in \mathscr{A} \qquad \text{and} \qquad R\rho(A) = \rho'(A)R, \quad A \in \mathscr{A}. \tag{4.1}$$

End $\mathscr{A}$ has a monoidal structure defined as follows: if $R \in (\rho, \rho')$ and $S \in (\sigma, \sigma')$ then

$$R \times S := R\rho(S) = \rho'(S)\, R \in (\rho\sigma, \rho'\sigma'). \tag{4.2}$$

ι will denote the identity automorphism of $\mathscr{A}$ so that (ι, ι) is just the centre of $\mathscr{A}$.[1]

If we restrict our attention to the powers of a single endomorphism ρ then as subsets of $\mathscr{A}$ we have

$$(\rho^r, \rho^s) \subset (\rho^{r+k}, \rho^{s+k}), \qquad (\rho^r, \rho^s)^* = (\rho^s, \rho^r), \quad k, r, s \in \mathbb{N}_0.$$

It follows (cf. the definition of ${}^0O_G$ in Sect. 1) that finite sums of intertwiners between powers of ρ form a ρ-stable $*$-subalgebra.

We pose the question of when the closure, which is a C^*-algebra with a distinguished endomorphism, is isomorphic to $\{O_G, \sigma_G\}$ for some G.

A closely related question is whether we can find a morphism μ: $O_G \to \mathscr{A}$ satisfying $\mu \circ \sigma = \rho \circ \mu$ and

$$\mu(\sigma^r, \sigma^s) \subset (\rho^r, \rho^s), \qquad r, s \in \mathbb{N}_0. \tag{4.3}$$

In this case we will speak of an *action* of O_G on $\mathscr{A}$.

From the way O_G has been defined an action of O_G on $\mathscr{A}$ is easily seen to be equivalent to an action M of $\mathscr{T}_G$ on $\mathscr{A}$. An action of a strict monoidal C^*-category $\mathscr{T}$ on $\mathscr{A}$ is a strict monoidal $*$-functor $M : \mathscr{T} \to$ End $\mathscr{A}$. This

[1] (ρ, ρ') is not just a complex linear space but an (ι, ι)-module in a natural way. We ignore this aspect here as we have in mind applications to C^*-algebras with trivial centre.

is a functor such that the induced maps $(\rho, \sigma) \to (M(\rho), M(\sigma))$ are linear, the unit of $\mathscr{T}$ is mapped onto the identity automorphism and

$$\begin{aligned} M(R^*) &= M(R)^* \\ M(\rho \otimes \rho') &= M(\rho)\, M(\rho') \\ M(R \otimes R') &= M(R) \times M(R'). \end{aligned} \tag{4.4}$$

If we take $\mathscr{T} = \mathscr{T}_G$, we get an action of a G-dual in the spirit of [8, Sect. 4] provided every irreducible representation is contained in some tensor power of the defining representation. This is, in particular, the case whenever $G \subset SU(H)$.

Here, it will be sufficient to look at actions from the point of view of the C^*-algebras O_G. Note that an action of O_G on $\mathscr{A}$ automatically gives us a unitary representation $p \to \varepsilon(p)$ of $\mathbb{P}_\infty$ in $\mathscr{A}$. We say that an endomorphism ρ has *permutation symmetry* if we can find such a representation with

$$\varepsilon(\sigma p) = \rho(\varepsilon(p)), \qquad p \in \mathbb{P}_\infty \tag{4.5}$$

$$\varepsilon(1, 1) \in (\rho^2, \rho^2) \tag{4.6}$$

$$\varepsilon(s, 1)X = \rho(X)\, \varepsilon(r, 1), \qquad X \in (\rho^r, \rho^s), \quad r, s \in \mathbb{N}_0. \tag{4.7}$$

These relations imply that $\varepsilon(p) \in (\rho^n, \rho^n)$ for $p \in \mathbb{P}_n$. Following our practice in Section 2 we will write ε for $\varepsilon(1, 1)$ so that we have

$$\varepsilon(r, 1) = \varepsilon\rho(\varepsilon) \cdots \rho^{r-1}(\varepsilon). \tag{4.8}$$

Equations (2.5) and (2.8) together with Theorem 3.5 show that the endomorphism σ_G of O_G has permutation symetry. The motivating examples for this concept arose in work on superselection structure in physics [9]. As we discuss in the Appendix, the endomorphism σ on $\mathbb{P}_\infty$ induces an endomorphism on $C^*(\mathbb{P}_\infty)$, also denoted by σ, with permutation symmetry. Furthermore, when the endomorphism ρ has permutation symetry given by ε, there is a canonical action of $C^*(\mathbb{P}_\infty)$ on $\mathscr{A}$ with $\mu \circ \sigma = \rho \circ \mu$ and $\mu(\theta) = \varepsilon$, where θ is now the unitary of $C^*(\mathbb{P}_\infty)$ corresponding to the transposition (12) in $\mathbb{P}_\infty$.

For a given permutation symmetry we have the problem of determining the kernel of the corresponding morphism $\mu: C^*(\mathbb{P}_\infty) \to \mathscr{A}$ and this amounts to giving the quasiequivalence class of the representation of each $\mathbb{P}_n$. Indeed, in the discussion following Lemma 3.6, we determined the ideal I_d for the permutation symmetry $p \to \theta(p)$ of the endomorphism σ_G on O_G. I_d is actually the ideal in $C^*(\mathbb{P}_\infty)$ generated by the totally antisymmetric projection in $C^*(\mathbb{P}_{d+1})$. We say that an endomorphism ρ has permutation symmetry of dimension d if we can choose the permutation symmetry $p \to \varepsilon(p)$ to have I_d as its associated kernel in $C^*(\mathbb{P}_\infty)$.

4.1. THEOREM. *If ρ is an endomorphism of a C^*-algebra $\mathscr{A}$ with permutation symmetry of dimension d realized by the unitary representation $p \to \varepsilon(p)$ of $\mathbb{P}_\infty$ in $\mathscr{A}$, there is a unique monomorphism $\mu\colon O_{U(d)} \to \mathscr{A}$ with $\mu(\theta) = \varepsilon$ and $\mu \circ \sigma = \rho \circ \mu$ defining an action of $O_{U(d)}$ on $\mathscr{A}$. If, furthermore, there is an $R \in (\iota, \rho^d)$ with*

$$RR^* = \sum_{p \in \mathbb{P}_d} \frac{1}{d!} \operatorname{sign}(p)\, \varepsilon(p)$$

then the action extends uniquely to $O_{SU(d)}$ if we require $\mu(S) = R$.

Proof. By definition, permutation symmetry of dimension d yields a monomorphism $\mu\colon O_{U(d)} \to \mathscr{A}$ with $\mu(\theta(p)) = \varepsilon(p)$, $p \in \mathbb{P}_\infty$ which then satisfies $\mu(\theta) = \varepsilon$ in particular and $\mu \circ \sigma = \rho \circ \mu$ by virtue of (4.5). To see that we actually get an action of $O_{U(d)}$ on $\mathscr{A}$, we need only recall what we have learned about the intertwiners of $O_{U(d)}$ in Theorem 3.5 and Lemma 3.6. We have seen that (σ^n, σ^n) is the linear span of the $\theta(p)$ with $p \in \mathbb{P}_n$, whilst $(\sigma^m, \sigma^n) = 0$ for $m \neq n$. If there is an $R \in (\iota, \rho^d)$ as above, then if $A \in O_{U(d)}$,

$$R\mu(A)R^* = \rho^d \circ \mu(A)\, RR^* = \mu \circ \sigma^d(A)\, \mu(SS^*) = \mu \circ \tau(A).$$

Here τ is as in Lemma 3.8, which guarantees that μ extends uniquely to a monomorphism $\mu\colon O_{SU(d)} \to \mathscr{A}$ with $\mu(S) = R$, $\mu(\sigma(S)) = \mu(\theta(d, 1)S) = \varepsilon(d, 1)R = \rho(R)$ so $\mu \circ \sigma = \rho \circ \mu$. Finally, to check that we still have an action, it sufficies, by Lemma 3.7, to show that $X \in (\sigma^r, \sigma^{r+kd})$ implies $\mu(X) \in (\rho^r, \rho^{r+kd})$. However, $X = XS^{k*}S^k$ so $\mu(X) = \mu(XS^{k*})R^k$. But $XS^{k*} \in (\sigma^{r+kd}, \sigma^{r+kd})$ as an element of $O_{U(d)}$ whilst $R^k \in (\rho^r, \rho^{r+kd})$ so $\mu(X) \in (\rho^r, \rho^{r+kd})$ completing the proof.

This theorem becomes more useful when we have a convenient criterion for deciding when a permutation symmetry has dimension d. The analysis of permutation symmetries is facilitated if our endomorphism admits a *left inverse* ϕ, i.e., a positive linear map $\phi\colon \mathscr{A} \to \mathscr{A}$ with $\phi(I) = I$ and

$$\phi(A\rho(B)) = \phi(A)B, \qquad A, B \in \mathscr{A}. \tag{4.9}$$

Note that $\rho \circ \phi$ is a conditional expectation from $\mathscr{A}$ to $\rho(\mathscr{A})$ and

$$\phi(A^*A) \geqslant \phi(A^*)\, \phi(A). \tag{4.10}$$

Furthermore if $X \in (\rho^r, \rho^s)$, then $\phi(X) \in (\rho^{r-1}, \rho^{s-1})$ $r, s \geqslant 1$ and the following lemma gives us a powerful inequality.

4.2. LEMMA. *If $X \in (\rho^r, \rho^s)$ then*

$$\phi(X^*X) \geqslant \varepsilon(1, r-1)\,\phi(\varepsilon)\,X^*X\phi(\varepsilon)\,\varepsilon(r-1, 1). \tag{4.11}$$

In particular

$$\|\phi(X^*X)\| \geqslant \|X\phi(\varepsilon)\|^2. \tag{4.12}$$

Proof. $X^*X = \varepsilon(1, r)\,\rho(X^*X)\,\varepsilon(r, 1)$ by (4.7). Taking $A = \rho(X)\,\varepsilon(r, 1)$ in (4.10) and using $\varepsilon(r, 1) = \varepsilon\rho(\varepsilon(r-1, 1))$ gives the desired result.

When $\mathscr{A}$ has trivial centre, this lemma can be used to prove that the self-adjoint operator $\phi(\varepsilon)$ has purely discrete spectrum. If ρ is irreducible, i.e., if (ρ, ρ) reduces to the complex numbers, then $\phi(\varepsilon) \in (\rho, \rho)$ is automatically a mutiple of the identity, $\phi((\varepsilon) = \lambda I$. We show that λ uniquely determines the kernel of the associated morphism $C^*(\mathbb{P}_\infty) \to \mathscr{A}$. As an example if we set

$$\phi(C) = \frac{1}{d} \sum_{i=1}^{d} \psi_i^* C \psi_i, \qquad C \in O_d \tag{4.13}$$

we get a left inverse for the canonical endomorphism σ on O_d. Here $\psi_1, \psi_2, \ldots, \psi_d$ denotes as usual the basic multiplet of isometries. A trivial computation shows that $\phi(\theta) = d^{-1}I$ which is the case of a permutation symmetry of dimension d, where the kernel is the ideal I_d. Note that $\phi(O_G) \subset O_G$ so that σ_G has a left inverse, too.

For a further example, notice that whenever $(\rho^r, \rho^s) = 0$ unless $r = s \bmod 2$, the permutation symmetry for ρ will certainly not be unique since we could replace $p \to \varepsilon(p)$ by $p \to \operatorname{sign}(p)\,\varepsilon(p)$ without violating (4.5), (4.6), or (4.7). Thus for $O_{U(d)}$ we could replace θ by $-\theta$ and it would still generate a permutation symmetry for $\sigma_{U(d)}$. This would give us an example where $\lambda = -d^{-1}$. The corresponding ideal in $C^*(\mathbb{P}_\infty)$ will be denoted I_{-d}. It is the ideal generated by the totally symmetric projection in $C^*(\mathbb{P}_{d+1})$. Finally, we show in the Appendix that $C^*(\mathbb{P}_\infty)$ yields an example with $\lambda = 0$, when the corresponding ideal is trivial.

This in fact exhausts all possibilities.

4.3. THEOREM. *If $p \to \varepsilon(p)$ gives a permutation symmetry for an endomorphism ρ of $\mathscr{A}$ and ϕ is a left inverse for ρ with $\phi(\varepsilon) = \lambda I$ then $\lambda \in \{0\} \cup \{\pm d^{-1} : d \in \mathbb{N}\}$. If $\lambda = \pm d^{-1}$, the kernel of the morphism $C^*(\mathbb{P}_\infty) \to \mathscr{A}$ determined by $p \to \varepsilon(p)$ is $I_{\pm d}$ whereas if $\lambda = 0$ we have a monomorphism $C^*(\mathbb{P}_\infty) \to \mathscr{A}$.*

Proof. The necessary computations can be found in Section 5 of [9] or Section 3 of [10] but will be repeated here for convenience. The strategy is as follows: we show that the image of $C^*(\mathbb{P}_\infty)$ is stable under ϕ. Iterating ϕ

then defines a state on $C^*(\mathbb{P}_\infty)$ and we compute the corresponding function ω_λ of positive type on $\mathbb{P}_\infty$ explicity

$$\omega_\lambda(p)I = \phi^\infty(\varepsilon(p)), \qquad p \in \mathbb{P}_\infty. \tag{4.14}$$

If $p \in \mathbb{P}_\infty$ and $p(1) = 1$ then $p = \sigma(p')$ so that

$$\phi(\varepsilon(p)) = \varepsilon(p'), \qquad p(1) = 1. \tag{4.15}$$

If $p(1) \neq 1$, write $p = (2p(1))(12)q$, where (st) denotes the transposition of s and t. Now $q(1) = 1$, so $p = \sigma(r')(12)\,\sigma(q')$ giving $\phi(\varepsilon(p)) = \varepsilon(r')\,\phi(\varepsilon)\,\varepsilon(q') = \lambda\varepsilon(r'q')$. Thus

$$\phi(\varepsilon(p)) = \lambda\varepsilon(p'), \qquad p(1) \neq 1, \tag{4.16}$$

where in (4.15) and (4.16), p' is the permutation obtained from p by deleting 1 from its cycle in the decomposition of p into disjoint cycles and then writing n for $n+1$ in this decomposition. Thus the image of $C^*(\mathbb{P}_\infty)$ is stable under ϕ. For n sufficiently large, $\phi^n(\varepsilon(p))$ will be a multiple of the identity; so (4.14) will define a function of positive type which we can compute explicitly from (4.15) and (4.16). ω_λ is a class function multiplicative on disjoint cycles and taking the value λ^{k-1} on a k-cycle. If E_n^s and E_n^a denote the totally symmetric and antisymmetric projections in the group algebra of $\mathbb{P}_n$ then

$$n!\,\omega_\lambda(E_n^s) = (1+\lambda)(1+2\lambda)\cdots(1+(n-1)\lambda)$$
$$n!\,\omega_\lambda(E_n^a) = (1-\lambda)(1-2\lambda)\cdots(1-(n-1)\lambda).$$

Thus the positivity condition is violated unless λ takes on one of the stated values. Now λ determines the kernel of the morphism $C^*(\mathbb{P}_\infty) \to \mathscr{A}$ since if $\lambda \neq 0$ it follows from (4.12) that if E is any projection in the group algebra of $\mathbb{P}_n$,

$$\varepsilon(E) = 0 \qquad \text{if and only if } \omega_\lambda(\varepsilon(E)) = 0.$$

On the other hand if $\lambda = 0$, ω_λ is the function of positive type defining the left regular representation and we have a monomorphism $C^*(\mathbb{P}_\infty) \to \mathscr{A}$.

Since ω_λ is even a class function on $\mathbb{P}_\infty$, the ω_λ define finite traces on $C^*(\mathbb{P}_\infty)$ which are faithful on the image of $C^*(\mathbb{P}_\infty)$ in $\mathscr{A}$.

As a corollary we get the following variant of Theorem 4.1.

4.4. COROLLARY. *Let $p \to \varepsilon(p)$ be a permutation symmetry for an endomorphism ρ of a C^*-algebra $\mathscr{A}$ and suppose there is an $R \in (\iota, \rho^d)$ with*

$$R^*\rho(R) = (-1)^{d-1} d^{-1} \cdot I \tag{4.17}$$

$$RR^* = d!^{-1} \sum_{p \in \mathbb{P}_d} \operatorname{sign}(p)\, \varepsilon(p), \tag{4.18}$$

then there is a unique monomorphism μ: $O_{SU(d)} \to \mathscr{A}$ with $\mu(\theta) = \varepsilon$, $\mu(S) = R$ and $\mu \circ \sigma = \rho \circ \mu$. μ defines an action of $O_{SU(d)}$ on $\mathscr{A}$.

Proof. Setting $\phi(A) = R^*\rho^{d-1}(A)R$, $A \in \mathscr{A}$ defines a left inverse for ρ. We now compute $\phi(\varepsilon)$. Since $\sigma^{d-1}(1, 1) = (1, d)(d-1, 1)$ in $\mathbb{P}_\infty$, we have $\phi(\varepsilon) = R^*\varepsilon(1, d)\, \varepsilon(d-1, 1)R$. Now by (4.18), $\varepsilon(d-1, 1)R = (-1)^{d-1}R$ and $R^*\varepsilon(1, d) = \rho(R^*)$ by (4.7) so by (4.17) $\phi(\varepsilon) = d^{-1}I$. By Theorem 4.3, ρ has permutation symmetry of dimension d so the result follows from Theorem 4.1.

Corollary 4.4 is used to prove Theorem 5 of [2]. If we add the hypothesis that $\mathscr{A}$ has trivial centre, the results announced in [2] show that for some closed subgroup $G \subset SU(d)$ unique up to conjugacy we have a monomorphism μ: $O_G \to \mathscr{A}$ extending the above action to O_G with the property that $\mu(\sigma^r, \sigma^s) = (\rho^r, \rho^s)$, $r, s \in \mathbb{N}_0$. This then identifies the C^*-algebra generated by intertwiners between powers of ρ as O_G [13].

Theorem 4.3 classifies permutation symmetry for irreducible endomorphisms with a left inverse on a unital C^*-algebra with trivial centre. The analysis can be pursued further; indeed the results obtained in the special context of superselection structure went further in several directions. As a stimulus to further research we point out that in this context $\lambda = \pm 1$ implies that ρ is an automorphism [9, Proposition 2.7], it proved possible to analyse the permutation symmetry of reducible endomorphisms ρ with a left inverse ϕ with $\phi(\varepsilon)$ invertible [9, Theorem 6.9] and then even to change the permutation symmetry so that ρ then had permutation symmetry of some dimension d characteristic of ρ [10, Sect. 5]. Furthermore the endomorphisms in question were shown to possess conjugates unique up to unitary equivalence generalizing the notion of inverse for an automorphism [11, Theorem 3.3].

Here is a first step towards a discussion of reducible endomorphisms.

4.5. LEMMA. *Let $\mathscr{A}$ be a unital C^*-algebra with trivial centre and ρ an endomorphism with left inverse ϕ and a permutation symmetry $p \to \varepsilon(p)$ then $\phi(\varepsilon)$ has purely discrete spectrum with zero as the only possible accumulation point.*

Proof. Cover the spectrum of $\phi(\varepsilon)$, a closed subset of $[-1, 1]$, by a finite number of intervals $[\lambda_0, \lambda_1)$, $[\lambda_1, \lambda_2), \ldots, [\lambda_{n-1}, \lambda_n)$ each of which contains a point of the spectrum in its interior. Then, by the functional calculus, we can find positive elements $a_0, a_1, ..., a_{k-1}$ in the C^*-algebra generated by $\phi(\varepsilon)$ and I such that $\sum_{i=0}^{n-1} a_i \leqslant 1$ and

$$\|\sqrt{a_i}\phi(\varepsilon)\| \geqslant \mu_i = \min\{|\lambda| : \lambda \in [\lambda_i, \lambda_{i-1}]\}.$$

However, since $a_i \in (\rho, \rho)$, $\phi(a_i) \in \mathbb{C}I$ because $\mathscr{A}$ has trivial centre thus by (4.12) we have $\phi(a_i) \geqslant \mu_i^2 I$, so

$$I \geqslant \phi\left(\sum_{i=0}^{n-1} a_i\right) \geqslant \sum_{i=0}^{n-1} \mu_i^2 I.$$

We have now actually proved that $\phi(\varepsilon)$ has discrete spectrum $\{\lambda_i, i \in \mathbb{N}\}$ say and that $\sum_{i \in \mathbb{N}} \lambda_i^2 \leqslant I$.

The next step involves examining the permutation symmetry induced on the subobjects of ρ, i.e., non-unital endomorphisms of the form $A \to \rho(A)E$. However, we will not pursue the matter further here as it falls outside the limited scope of this paper. The context here is too narrow in several respects. One should probably look at endomorphisms of $\mathscr{A}$ into matrix algebras over $\mathscr{A}$ which do not necessarily preserve the unit so as to have a category closed under subobjects and direct sums. It might even be advisable to work in the context of $\mathscr{A}$-bimodules. Furthermore permutation symmetry should be looked at not for a single endomorphism but directly for a monoidal category closed under subobjects and direct sums.

Appendix

The C^*-algebra $C^*(\mathbb{P}_\infty)$ of the group $\mathbb{P}_\infty$ of finite permutations of the integers deserves attention in the context of permutation symmetry. $C^*(\mathbb{P}_\infty)$ is the inductive limit of the finite-dimensional C^*-algebras $C^*(\mathbb{P}_n)$.

We have a canonical unitary representation of $\mathbb{P}_\infty$ in $C^*(\mathbb{P}_\infty)$ which we will denote by $p \to \theta(p)$ and we will continue to adopt the same notational conventions as in Section 2 writing, for example, θ to denote $\theta(1, 1)$. The canonical endomorphism σ on $\mathbb{P}_\infty$ which shifts to the right induces an endomorphism on $C^*(\mathbb{P}_\infty)$ which will again be denoted by σ so that

$$\theta(\sigma p) = \sigma(\theta(p)), \qquad p \in \mathbb{P}_\infty.$$

A.1. Lemma. *If $X \in C^*(\mathbb{P}_\infty)$ then*

$$\sigma(X) = \lim_{k \to \infty} \theta(k, 1)\, X\theta(k, 1)^*.$$

Proof. If $X \in C^*(\mathbb{P}_n)$ then $\sigma(X) = \theta(k, 1)\, X\theta(k, 1)^*$, $k \geqslant n$.

A.2. COROLLARY. *Let* $X \in (\sigma^r, \sigma^r)$ *then*

$$\theta(r, 1)X = \sigma(X)\, \theta(r, 1).$$

Proof. $\theta(k, 1) = \theta\sigma(\theta) \cdots \sigma^{k-1}(\theta)$, hence if $k \geqslant r$,

$$\theta(k, 1)\, X\theta(k, 1)^* = \theta(r, 1)\, X\theta(r, 1)^*;$$

and proceeding to the limit $k \to \infty$ we get the result.

A.3. LEMMA. $(\sigma^r, \sigma^s) = 0$ *if* $r \neq s$.

Proof. Suppose for concreteness $s > r$, say $s = r + k$. Let $X \in (\sigma^r, \sigma^{r+k})$. Given $\varepsilon > 0$, we can find $n \in \mathbb{N}$ and an $X' \in C^*(\mathbb{P}_n)$ with $\|X - X'\| < \varepsilon$. Without loss of generality, we may suppose $r \geqslant n$ then

$$\begin{aligned}\|(\sigma^r(\theta) - \sigma^{r+k}(\theta))X'\| &\leqslant \|(\sigma^r(\theta) - \sigma^{r+k}(\theta))X\| + 2\varepsilon \\ &\leqslant \|\sigma^r(\theta)X - X\sigma^r(\theta)\| + 2\varepsilon.\end{aligned}$$

Since $X' \in C^*(\mathbb{P}_n)$, it commutes with $\sigma^r(\theta)$ so $\|(\sigma^r(\theta) - \sigma^{r+k}(\theta))X'\| < 4\varepsilon$. Now X' and $\sigma^r(\theta) - \sigma^{r+k}(\theta)$ refer to different "variables": if we consider $\mathbb{P}_n \times \mathbb{P}_m$ as a subgroup of $\mathbb{P}_{n+m}$ in the natural way then $C^*(\mathbb{P}_n \times \mathbb{P}_m) = C^*(\mathbb{P}_n) \otimes C^*(\mathbb{P}_m) \subset C^*(\mathbb{P}_{n+m}) \subset C^*(\mathbb{P}_\infty)$, so

$$\|(\sigma^r(\theta) - \sigma^{r+k}(\theta))X'\| = \|\sigma^r(\theta) - \sigma^{r+k}(\theta)\|\, \|X'\| < 4\varepsilon.$$

Since $\|X\|$ is majorized by a multiple of ε, we conclude that $X = 0$.

The last two results actually show that:

A.4. THEOREM. *The endomorphism* σ *on* $C^*(\mathbb{P})$ *has permutation symmetry.*

However, we can also show that we have just the expected intertwiners for the powers of σ.

A.5. LEMMA. $(\sigma^r, \sigma^r) = C^*(\mathbb{P}_r)$, $r \in \mathbb{N}$.

Proof. Given $X \in (\sigma^r, \sigma^r)$, we want to conclude that $X \in C^*(\mathbb{P}_r)$. Now we have a conditional expectation $m_r\colon C^*(\mathbb{P}_\infty) \to C^*(\mathbb{P}_r)$ defined by restricting elements in the group algebra of $\mathbb{P}_\infty$ to the subgroup $\mathbb{P}_r$. Replacing X by $X - m_r(X)$, we may suppose $m_r(X) = 0$. Given $\varepsilon > 0$, we can find $n \in \mathbb{N}$ and an $X' \in C^*(\mathbb{P}_n)$ with $m_r(X') = 0$ and $\|X - X'\| < \varepsilon$. Without loss of generality, we may even suppose that X and X' are self-adjoint. Now

$\sigma^n(X') = \theta(p) X' \theta(p)^{-1}$, where $p \in \mathbb{P}_\infty$ can be chosen such that $p(i) = i$, $i = 1, 2, \ldots, r$, since $m_r(X') = 0$. But this implies that $X = \theta(p) X \theta(p)^{-1}$ as $X \in (\sigma^r, \sigma^r)$. Hence $\|X' - \sigma^n(X')\| < 2\varepsilon$. Now $X' - \sigma^n(X')$ is of the form $Y \otimes I - I \otimes Y$ in $C^*(\mathbb{P}_n) \otimes C^*(\mathbb{P}_n)$ so Y must be close to a multiple of the identity. However, as $m_r(X') = 0$, we have $\|X'\| < 2\varepsilon$ and thus $\|X\| < 3\varepsilon$. Hence $X = 0$ completing the proof.

If ρ is an endomorphism of a C^*-algebra $\mathscr{A}$ with a permutation symmetry $p \to \varepsilon(p)$ then as discussed in Section 4 we have a canonical morphism $\mu \colon C^*(\mathbb{P}_\infty) \to \mathscr{A}$ with $\mu(\theta) = \varepsilon$ and $\mu \circ \sigma = \rho \circ \mu$. It follows from Lemmas A.3 and A.5 that $\mu(\sigma^r, \sigma^s) \subset (\rho^r, \rho^s)$ so that we in fact get an action of $C^*(\mathbb{P}_\infty)$ on $\mathscr{A}$.

Lemma A.5 also implies that $(\sigma, \sigma) = \mathbb{C}I$ so that the endomorphism σ is irreducible. The computations in Theorem 4.2 imply that a left inverse ϕ for σ must satisfy $\phi(\theta) = 0$ and, since $C^*(\mathbb{P}_\infty)$ is generated by $p \to \theta(p)$, must be unique.

A.6. LEMMA. *The endomorphism σ on $C^*(\mathbb{P}_\infty)$ has a unique left inverse ϕ and $\phi(\theta) = 0$.*

Proof. We need only prove the existence of ϕ which we do by defining ϕ to be the inductive limit of the mappings $\phi_n \colon C^*(\mathbb{P}_{n+1}) \to C^*(\mathbb{P}_n)$, where

$$\phi_n(X) = m_n(\theta(1, n) X \theta(1, n)^*), \qquad X \in C^*(\mathbb{P}_{n+1}).$$

Now suppose that $\mathscr{A}$ is the quotient of $C^*(\mathbb{P}_\infty)$ by some closed ideal I and use $\mathscr{A}_n$ to denote the image of $C^*(\mathbb{P}_n)$ and $\theta(p)$ the image of $\theta(p)$ in $\mathscr{A}$. Then the canonical endomorphism σ passes to the quotient and the analogues of Lemma A.1 and Corollary A.2 hold. One might be tempered on this scanty evidence to conjecture that $p \to \theta(p)$ is always a permutation symmetry for σ regarded as an endomorphism of $\mathscr{A}$. As this question is relevant to the classification of permutation symmetries we give here some examples. The ideals I involved will be specified by giving minimal central projections in subalgebras $C^*(\mathbb{P}_n)$ which generate the ideal. The minimal projections in $C^*(\mathbb{P}_n)$ are in 1 to 1-correspondence with the Young diagrams with n squares. So the examples will be described by a generating set of Young diagrams. Thus a single column of length $d+1$ gives $O_{U(d)}$ as the quotient of $C^*(\mathbb{P}_\infty)$ by the ideal generated by the totally antisymmetric projection in $C^*(\mathbb{P}_{d+1})$. Here we have permutation symmetry.

More generally any rectangular Young diagram gives us an example with permutation symmetry. In this case $\mathscr{A}$ can still be realized as a C^*-subalgebra of O_d with the endomorphism σ being induced by the canonical Hilbert space in O_d. We need only redefine θ within O_d by setting

$$\theta = \sum_{i,j} \operatorname{sign}(i, j)\, \psi_i \psi_j \psi_i^* \psi_j^*,$$

where $\text{sign}(i, j) = -1$ if $i, j > n$ and $+1$ otherwise and let $\mathscr{A}$ be the σ-stable C^*-subalgebra of O_d generated by θ. The ideal now corresponds to a rectangular Young diagram whose column length is $n+1$ and whose row length is $d-n+1$. Since $\mathscr{A}_n \subset (H^n, H^n)$, the C^*-subalgebra of $\mathscr{A}$ generated by $\mathscr{A}_n$ and $\sigma^n(\mathscr{A}_m)$ is therefore canonically isomorphic to $\mathscr{A}_n \otimes \mathscr{A}_m$: no more is needed in the proof of Lemma A.3. Permutation symmetries of this type are well known from the theory of superselection structures; cf. [9, Theorem 6.9].

The final example is instructive: let e_n, $n \in \mathbb{N}$ be the orthonormal basis of a Hilbert space and let $p \to \theta(p)$ be the unitary representation of $\mathbb{P}_\infty$ defined by $\theta(p)e_n = e_{p(n)}$. Let $\mathscr{A}$ be the C^*-algebra generated by $p \to \theta(p)$. The following assertions can be easily verified so the details will be left to the reader. $\mathscr{A}$ is irreducible, in fact $\mathscr{A}$ is just the compact operators with the multiples of the identity adjoined. The ideal in $C^*(\mathbb{P}_\infty)$ is here generated by two Young diagrams: a single column of length 3 and a 2×2-square. $X \in (\sigma^r, \sigma^s)$ if and only if $(e_n, Xe_m) = 0$ for $m > r$ and for $n > s$. Thus (σ^r, σ^s) is not only non-zero for $r \neq s$ in contrast to Lemma A.3 but includes (σ^r, σ^r) if $s > r$. Nevertheless $p \to \theta(p)$ is a permutation symmetry for σ.

This example also illustrates an obstruction to finding a left inverse ϕ with $\phi(\theta)$ invertible. Let E denote the projection onto $\mathbb{C}e_1$ then $E \in (\sigma, \sigma)$ so σ is reducible. $\sigma(E)$ is the projection onto $\mathbb{C}e_2$ so $E\sigma(E) = 0$. Applying ϕ to this equation and noting that $\phi(E) \in (\iota, \iota) = \mathbb{C}I$ we conclude that $\phi(E) = 0$. But now by (4.12), $E\phi(\theta) = 0$ so that $\phi(\theta)$ is not invertible. In fact σ has a unique left inverse given by $\phi(A) = U^*AU$, where U is the one-sided shift, $Ue_n = e_{n+1}$, and $\phi(\theta) = I - E$.

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